SL Paper 1

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y^3 - x^3$ for which $y = 1$ when $x = 0$. Use Euler's method with a step length of 0.1 to find an approximation for the value of y when $x = 0.4$.

Use the integral test to determine whether or not $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

Let

$I_n = \int_1^{\infty} x^n \mathrm{e}^{-x} \mathrm{d}x$ where $n \in \mathbb{N}$.

a. Using l'Hôpital's rule, show that [4]

$\lim\limits_{x \to \infty} x^n \mathrm{e}^{-x} = 0$ where $n \in \mathbb{N}$.

b.i. Show that, for $n \in \mathbb{Z}^+$, [4]

$$I_n = \alpha \mathrm{e}^{-1} + \beta n I_{n-1}$$

where α, β are constants to be determined.

b.ii. Determine the value of I_3, giving your answer as a multiple of e^{-1}. [5]

Given that y is a function of x, the function z is given by $z = \frac{y-x}{y+x}$, where $x \in \mathbb{R},\ x \neq 3,\ y + x \neq 0$.

a. Show that $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{2}{(y+x)^2}\left(x\frac{\mathrm{d}y}{\mathrm{d}x} - y\right)$. [3]

b. Show that the differential equation $f(x)\left(x\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = y^2 - x^2$ can be written as $f(x)\frac{\mathrm{d}z}{\mathrm{d}x} = 2z$. [2]

c. Hence show that the solution to the differential equation $(x-3)\left(x\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = y^2 - x^2$ given that $x = 4$ when $y = 5$ is $\frac{y-x}{y+x} = \left(\frac{x-3}{3}\right)^2$. [7]

Find the interval of convergence of the series $\sum\limits_{k=1}^{\infty} \frac{(x-3)^k}{k^2}$.

The function $f:\mathbb{R}\to\mathbb{R}$ is defined by $f: x \to \begin{cases} -3x+1 & \text{for } x<0 \\ 1 & \text{for } x=0 \\ 2x^2-3x+1 & \text{for } x>0 \end{cases}$.

By considering limits prove that f is

a. continuous at $x=0$; [4]

b. differentiable at $x=0$. [5]

Consider the infinite series $S=\sum\limits_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$.

a. Show that the series is conditionally convergent but not absolutely convergent. [6]

b. Show that $S>0.4$. [2]

The function f is defined by

$$f(x)=\frac{e^x+e^{-x}+2\cos x}{4},\ x\in\mathbb{R}.$$

The random variable X has a Poisson distribution with mean μ.

a.i. Show that $f^{(4)}x=f(x)$; [4]

a.ii. By considering derivatives of f, determine the first three non-zero terms of the Maclaurin series for $f(x)$. [4]

b.i. Write down a series in terms of μ for the probability $p=\mathrm{P}[X\equiv 0(\bmod 4)]$. [2]

b.ii. Show that $p=e^{-\mu}f(\mu)$. [1]

b.iii. Determine the numerical value of p when $\mu=3$. [2]

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a. Find the general solution of the differential equation $(1-x^2)\frac{dy}{dx}=1+xy$, for $|x|<1$. [7]

b. (i) Show that the solution $y=f(x)$ that satisfies the condition $f(0)=\frac{\pi}{2}$ is $f(x)=\frac{\arcsin x+\frac{\pi}{2}}{\sqrt{1-x^2}}$. [6]

(ii) Find $\lim\limits_{x\to -1} f(x)$.

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a. Calculate the following limit [3]

$\lim_{x \to 0} \frac{2^x - 1}{x}$.

b. Calculate the following limit [5]

$\lim_{x \to 0} \frac{(1+x^2)^{\frac{3}{2}} - 1}{\ln(1+x) - x}$.

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a. By evaluating successive derivatives at $x = 0$, find the Maclaurin series for $\ln \cos x$ up to and including the term in x^4 . [8]

b. Consider $\lim_{x \to 0} \frac{\ln \cos x}{x^n}$, where $n \in \mathbb{R}$. [5]

Using your result from (a), determine the set of values of n for which

(i) the limit does not exist;

(ii) the limit is zero;

(iii) the limit is finite and non-zero, giving its value in this case.

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a. Given that the series $\sum_{n=1}^{\infty} u_n$ is convergent, where $u_n > 0$, show that the series $\sum_{n=1}^{\infty} u_n^2$ is also convergent. [4]

b.i. State the converse proposition. [1]

b.ii. By giving a suitable example, show that it is false. [1]

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b. (i) Sum the series $\sum_{r=0}^{\infty} x^r$. [11]

(ii) **Hence**, using sigma notation, deduce a series for

(a) $\frac{1}{1+x^2}$;

(b) $\arctan x$;

(c) $\frac{\pi}{6}$.

c. Show that $\sum_{n=1}^{100} n! \equiv 3(\bmod 15)$. [4]

(a) Assuming the Maclaurin series for e^x, determine the first three non-zero terms in the Maclaurin expansion of $\frac{e^x - e^{-x}}{2}$.

(b) The random variable X has a Poisson distribution with mean μ. Show that $P(X \equiv 1(\bmod 2)) = a + be^{c\mu}$ where a, b and c are constants whose values are to be found.

Use l'Hôpital's rule to find $\lim\limits_{x \to 0}(\csc x - \cot x)$.

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. Using l'Hôpital's Rule, determine the value of [6]

$$\lim_{x \to 0} \frac{\tan x - x}{1 - \cos x}.$$

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a. (i) Find the range of values of n for which $\int_1^\infty x^n \mathrm{d}x$ exists. [7]

(ii) Write down the value of $\int_1^\infty x^n \mathrm{d}x$ in terms of n , when it does exist.

b. Find the solution to the differential equation [8]

$$(\cos x - \sin x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\cos x + \sin x)y = \cos x + \sin x \ ,$$

given that $y = -1$ when $x = \frac{\pi}{2}$.

The function f is defined by $f(x) = \mathrm{e}^x \cos x$.

a. Show that $f''(x) = -2\mathrm{e}^x \sin x$. [2]

b. Determine the Maclaurin series for $f(x)$ up to and including the term in x^4 . [5]

c. By differentiating your series, determine the Maclaurin series for $\mathrm{e}^x \sin x$ up to the term in x^3 . [4]

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a. Differentiate the expression $x^2 \tan y$ with respect to x, where y is a function of x. [3]

b. Hence solve the differential equation $x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + x \sin 2y = x^3 \cos^2 y$ given that $y = 0$ when $x = 1$. Give your answer in the form $y = f(x)$. [7]

Solve the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \sqrt{1 + x^2}$

given that $y = 1$ when $x = \sqrt{3}$.

Consider the infinite series $S = \sum_{n=1}^{\infty} \frac{x^n}{2^{2n}(2n^2-1)}$.

(a) Determine the radius of convergence.

(b) Determine the interval of convergence.

Solve the following differential equation

$$(x+1)(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} + y = x + 1$$

giving your answer in the form $y = f(x)$.

Given that $\frac{\mathrm{d}x}{\mathrm{d}y} + 2y\tan x = \sin x$, and $y = 0$ when $x = \frac{\pi}{3}$, find the maximum value of y.
